

SPLICING FOR MOTIVIC ZETA FUNCTIONS

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ABSTRACT. We lift the splicing formula of Némethi and Veys, which deals with polynomials in two variables, to the motivic level. After defining the motivic zeta function and the monodromic motivic zeta function with respect to a differential form, we prove a splicing formula for them, which specializes to this formula of Némethi and Veys. We also show that we cannot introduce a monodromic motivic zeta functions in terms of a (splice) diagram since it does not contain all the necessary information. In the last part we discuss the generalized monodromy conjecture of Némethi and Veys. The statement also holds for motivic zeta functions but it turns out that the analogous statement for monodromic motivic zeta functions is not correct. We show some examples illustrating this.

INTRODUCTION

We will consider here a polynomial $f \in \mathbb{C}[x, y]$, which has at most a singularity at the origin.

Splicing. The topology of a plane curve singularity $\{f = 0\}$ at the origin is closely related to its link $\{(z_1, z_2) \in \mathbb{C}^2 \mid f(z_1, z_2) = 0, |z_1|^2 + |z_2|^2 = \varepsilon\}$, where ε is sufficiently small. This link can be studied by looking at its splice diagram.

Splicing itself is originally a technique from link theory which constructs a new link out of two given links. We can also use splicing to decompose complicated links into easier links. This decomposition procedure has a nice description for our case of plane curve singularities. Fix an embedded resolution of singularities $\pi : X \rightarrow \mathbb{A}_{\mathbb{C}}^2$ of $f^{-1}(0)$. The splice diagram Γ associated to f and π is then constructed by taking the dual graph of the exceptional curves of π , removing nodes if necessary, where these nodes correspond to exceptional curves, and adding decorations to the edges.

The splicing of a splice diagram Γ along an edge e produces two new splice diagrams Γ_R and Γ_L . As shown in Figure 1, it divides Γ into two pieces and then makes them again into a splice diagram by adding appropriate multiplicities M and M' . Splicing the links of Γ_L and Γ_R together, we obtain the link of Γ (see [12] for more details).

Némethi and Veys in [18] applied this splicing technique to the topological zeta function.

Topological zeta functions. Denef and Loeser introduced the topological zeta function in [8] as follows: let $E_j, j \in J$, be the irreducible components of $\pi^{-1}(f^{-1}(0))$, N_j the multiplicity of

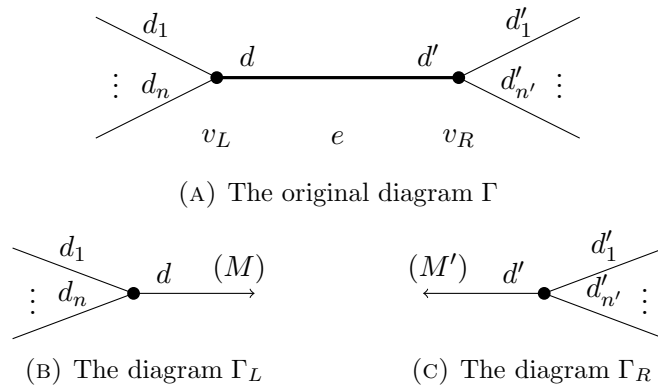


FIGURE 1. Splicing a diagram Γ along an edge e .

π^*f along E_j in and $\nu_j - 1$ the multiplicity of $\pi^*(dx \wedge dy)$ along E_j . The (local) topological zeta function is then

$$(1) \quad Z_f^{\text{top}}(s) = \sum_{\emptyset \neq I \subseteq J} \chi_{\text{top}}(E_I^\circ \cap \pi^{-1}(0)) \prod_{i \in I} \frac{1}{N_i s + \nu_i} \in \mathbb{Q}(s),$$

where $E_I^\circ = \cap_{i \in I} E_i \setminus (\cup_{i \in J \setminus I} E_i)$. Using the existence of a minimal resolution, it is obvious that this is independent of the chosen resolution. In general dimension the independence was originally proven by a limit argument using the p -adic zeta functions $\int_{p\mathbb{Z}_p^2} |f|_p^s |dx|$, where we assumed that $f \in \mathbb{Q}[x, y]$. This can also be shown by using the weak factorization theorem [25][1] or by considering the topological zeta function as a specialization of the motivic zeta function, which we will discuss further on.

To obtain a splicing formula, Némethi and Veys incorporated a differential form ω into the splice diagram and defined a topological zeta function $Z_{f,\omega}^{\text{top}}(s) = Z_\Gamma^{\text{top}}(s)$ for such splice diagrams (and thus with respect to ω) by using (1), where we redefine ν_i in terms of ω . This corresponds to considering the p -adic zeta functions $\int_{p\mathbb{Z}_p^2} |f|_p^s |\omega|$, given $f \in \mathbb{Q}[x, y]$. These zeta functions with respect to a differential form were already introduced in [2], [3] and [21] with a restriction on the support of ω . In [22] and [18] there is no restriction on the support but only the topological zeta function is considered.

Splicing formula and generalized monodromy conjecture. Némethi and Veys showed then that there is a nice splicing formula connecting the involved diagrams in Figure 1 and their topological zeta functions:

$$(2) \quad Z_\Gamma^{\text{top}}(s) = Z_{\Gamma_L}^{\text{top}}(s) + Z_{\Gamma_R}^{\text{top}}(s) - \frac{1}{(Ms + i)(M's + i')}.$$

These i and i' are also introduced when we splice Γ and the involved zeta functions are with respect to some differential form, whose information is contained in the diagram. They used this to prove the generalized monodromy conjecture, which predicts the existence of a class of ‘allowed’ differential forms such that the following holds:

- for every allowed form ω and every pole s_0 of $Z_{f,\omega}^{\text{top}}(s)$, $\exp(2\pi i s_0)$ is a local monodromy eigenvalue of f ;
- $dx \wedge dy$ is allowed;
- every local monodromy eigenvalue of f is obtained as a pole of $Z_{f,\omega}^{\text{top}}(s)$ for some allowed ω .

Motivic zeta functions. We consider here the Grothendieck ring of varieties $K_0(\text{Var}_{\mathbb{C}})$, its localization $\mathcal{M}_{\mathbb{C}}$ with respect to $\mathbb{L} = [\mathbb{A}_{\mathbb{C}}^1]$ and the completion $\hat{\mathcal{M}}_{\mathbb{C}}$. We denote by $[X]$ the class of the variety X . Also $\mathcal{L}_n(X)$ will be the scheme of n -jets of a variety X and $\mathcal{L}(X)$ will be the arc space of X . All of this will be introduced in more detail in Section 1. As mentioned before, the topological zeta function can also be considered as an avatar of the (local) motivic zeta function $Z_f(T) \in \mathcal{M}_{\mathbb{C}}[[T]]$. This motivic zeta function is defined by $Z_f(T) := \sum_{n>0} [\mathfrak{X}_n] \mathbb{L}^{-dn} T^n \in \mathcal{M}_{\mathbb{C}}[[T]]$ where $\mathfrak{X}_n := \{\varphi \in \mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^2) \mid \text{ord}_t f_n(\varphi) = n, \pi_0^n(\varphi) = 0\}$. There is also an explicit formula for $Z_f(T)$ in terms of a log resolution similar to the one in (1).

We introduce here a motivic zeta function with respect to a differential form. We do this by observing that $[\mathfrak{X}_n] \mathbb{L}^{-dn}$ is the (naive) measure of a cylinder \mathcal{Z}_n in the arc space $\mathcal{L}(\mathbb{A}_{\mathbb{C}}^2)$. After defining a new motivic measure μ_ω with values in $\hat{\mathcal{M}}_{\mathbb{C}}$, we obtain a motivic zeta function $Z_{f,\omega}(T) := \sum_{n>0} \mu_\omega(\mathcal{Z}_n) T^n \in \hat{\mathcal{M}}_{\mathbb{C}}[[T]]$. For this motivic zeta function $Z_{f,\omega}(T)$ there exists again a formula:

$$(3) \quad Z_{f,\omega}(T) = \sum_{\emptyset \neq I \subseteq J} (\mathbb{L} - 1)^{|I|} [E_I^\circ \cap \pi^{-1}(0)] \prod_{i \in I} \frac{T^{N_i}}{\mathbb{L}^{v_i} - T^{N_i}} \in \hat{\mathcal{M}}_{\mathbb{C}}[[T]].$$

We want to extend the definition of the motivic zeta function to the case of splice diagrams, but it turns out to be a bit hard. To ease our notation and to simplify our proofs, we introduce

the notion of a *diagram*: this is actually the same as a splice diagram except we can choose how many nodes (of valency two) we remove (instead of all of them). A diagram where no nodes were removed is called realizable and refining a diagram is adding nodes again to the diagram to obtain a new diagram. The definition in (3) for realizable diagrams then extends to all diagrams by choosing any refinement.

This leads us to our theorem.

Theorem. *Consider a diagram Γ and the splicing of Γ into Γ_L and Γ_R . Then we have*

$$Z_\Gamma(T) = Z_{\Gamma_L}(T) + Z_{\Gamma_R}(T) - \frac{(\mathbb{L} - 1)^2 T^{M+M'}}{(\mathbb{L}^i - T^M)(\mathbb{L}^{i'} - T^{M'})}.$$

This formula specializes to (2).

Splice diagrams and monodromic motivic zeta functions. Consider $\hat{\mu} = \varprojlim_n \mu_n$, where μ_n is the group of n -th roots of unity and the localized monodromic Grothendieck ring $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. This $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ will be defined in more detail in Section 1. The next step is to add a $\hat{\mu}$ -action to the motivic zeta function, which is done by considering the monodromic motivic zeta function $Z_f^{\hat{\mu}}(T) := \sum_{n>0} [\mathfrak{X}_{n,1}] \mathbb{L}^{-dn} T^n \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}[[T]]$, where $\mathfrak{X}_{n,1} := \{\varphi \in \mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^2) \mid f_n(\varphi) \equiv t^n \pmod{(t^{n+1})}, \pi_0^n(\varphi) = 0\}$. The action of μ_n on $\mathfrak{X}_{n,1}$ is defined by $a \cdot \varphi(t) = \varphi(at)$, where $a \in \mu_n$ and $\varphi \in \mathcal{L}_m(\mathbb{A}_{\mathbb{C}}^n)$, which induces an action of $\hat{\mu}$.

In section 1 we define $Z_{f,\omega}^{\hat{\mu}}(T)$, which is the monodromic motivic zeta function with respect to a differential form ω . It turns out that we cannot define $Z_{f,\omega}^{\hat{\mu}}(T)$ in terms of a (splice) diagram: we will show that there exist $\lambda, \lambda' \in \mathbb{C} \setminus \{0, 1\}$ such that

$$Z_{f_\lambda}^{\hat{\mu}}(T) \neq Z_{f_{\lambda'}}^{\hat{\mu}}(T),$$

where $f_\lambda = xy^2(x-y)(x-\lambda y) \in \mathbb{C}[x, y]$. For this λ and λ' , we find that the associated splice diagrams are the same but the monodromic motivic zeta functions are not. Hence we cannot define $Z_{f,\omega}^{\hat{\mu}}(T)$ in terms of a diagram.

To construct such λ and λ' we use the Picard morphism constructed by Ekedahl in [13]. In the appendix we prove and explain the details of this construction since [13] was never published.

Generalized monodromy conjecture for motivic zeta functions. The straightforward generalization to the motivic zeta functions turns out to be true. We will specify what we mean by a pole in Section 5.

Corollary. *Consider the set of allowed forms for a diagram Γ of $f \in \mathbb{C}[x, y]$. It satisfies the following conditions:*

- *for every allowed form ω , every pole of $Z_{f,\omega}(T)$ induces a monodromy eigenvalue. More specifically Theorem 5.1 holds.*
- *$dx \wedge dy$ is allowed;*
- *every monodromy eigenvalue is obtained as a pole of the motivic zeta function of f with respect to ω .*

However in the case of the monodromic motivic zeta function we show that this does not hold. First we give an easy example where an allowed form gives a non-desired pole of the twisted topological zeta function. The twisted topological zeta function will also be introduced in Section 1. This means that it must occur as a pole of the monodromic motivic zeta function and thus the generalized monodromy conjecture cannot be valid. Secondly we produce an example which shows that a subset of allowed forms is not sufficient: there exists no allowed differential form such that all the poles of the monodromic motivic zeta function induce monodromy eigenvalues and such that a particular monodromy eigenvalue is obtained.

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1. MOTIVIC ZETA FUNCTIONS AND GROTHENDIECK RINGS

In this section we introduce the necessary Grothendieck rings and zeta functions. A variety will be a complex algebraic variety, i.e. a reduced separated scheme of finite type over \mathbb{C} . The associated category will be denoted by $\text{Var}_{\mathbb{C}}$. Also fix a polynomial $f \in \mathbb{C}[x_1, \dots, x_d]$, which we will also consider as a morphism $\mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^1$. See [10] and [9] for more background information.

1.1. Grothendieck rings. The Grothendieck ring $K_0(\text{Var}_{\mathbb{C}})$ of complex varieties is the abelian group generated by isomorphism classes of varieties, where we denote the class of a variety X by $[X]$, subject to the relations

$$[X] = [X \setminus Z] + [Z], \text{ where } X \text{ is a variety and } Z \text{ is a closed subvariety of } X.$$

The ring structure is induced by defining $[X] \cdot [Y] = [X \times Y]$ for varieties X and Y . We denote by \mathbb{L} the class of $\mathbb{A}_{\mathbb{C}}^1$ and the localization $K_0(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ is denoted by $\mathcal{M}_{\mathbb{C}}$. Consider for every $i \in \mathbb{N}$ the subgroup F^i of $\mathcal{M}_{\mathbb{C}}$ generated by the elements $\frac{[X]}{\mathbb{L}^n}$, where X is a variety and $i \in \mathbb{N}$ such that $\dim X - n \leq -i$. These subgroups form a descending filtration on $\mathcal{M}_{\mathbb{C}}$ and its completion $\varprojlim_n (\mathcal{M}_{\mathbb{C}}/F^n)$ is denoted by $\hat{\mathcal{M}}_{\mathbb{C}}$.

The group of n -th roots of unity is denoted by μ_n . A *good* μ_n -action on a variety X is an algebraic group action $\mu_n \times X \rightarrow X$ such that each orbit is contained in an affine open subvariety. Consider the group $\hat{\mu} := \varprojlim_n \mu_n$. A *good* $\hat{\mu}$ -action on X is an action of $\hat{\mu}$ on X which factors through a good μ_n -action for some $n \in \mathbb{N}$.

Call two $\hat{\mu}$ -actions on varieties X and Y isomorphic if there is an isomorphism between X and Y which also preserves the $\hat{\mu}$ -action. The monodromic Grothendieck ring of complex varieties with good $\hat{\mu}$ -action $K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}})$ is defined as the abelian group generated by isomorphism classes of varieties with a good $\hat{\mu}$ -action, where the class of a variety X with action $\alpha : \hat{\mu} \times X \rightarrow X$ is denoted by $[X, \alpha]$ or $[X]$, subject to the relations

- $[X] = [X \setminus Z] + [Z]$, where X is a variety with $\hat{\mu}$ -action, Z is a closed subvariety invariant under the action, and the actions on $X \setminus Z$ and Z are induced by the one on X ;
- $[V] = [X \times \mathbb{A}_{\mathbb{C}}^n]$, where $V \rightarrow X$ is vector bundle of rank n over a variety X with a $\hat{\mu}$ -action which is linear over the action on X . (We are following [5] here)

The ring structure can be defined in the same way as before. Again we denote by \mathbb{L} the class of $\mathbb{A}_{\mathbb{C}}^1$, where we equip it with the trivial $\hat{\mu}$ -action and we define $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ to be the localization $K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$. Analogously for every $i \in \mathbb{N}$ we define \hat{F}^i as the subgroup generated by the elements $\frac{[X]}{\mathbb{L}^n}$ where X is a variety such that $\dim X - n \leq -i$. Its completion $\varprojlim_n (\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}/\hat{F}^n)$ is the monodromic completed Grothendieck ring $\hat{\mathcal{M}}_{\mathbb{C}}^{\hat{\mu}}$.

1.2. n -jets and arcs. Fix a variety X of dimension d in this subsection and let $n \in \mathbb{N}$. Recall that the functor

$$\text{Sch}_{\mathbb{C}} \rightarrow \text{Set} : Y \mapsto X \left(Y \times_{\mathbb{C}} \text{Spec} \left(\frac{\mathbb{C}[t]}{(t^{n+1})} \right) \right) = \text{Hom}_{\mathbb{C}} \left(Y \times_{\mathbb{C}} \text{Spec} \left(\frac{\mathbb{C}[t]}{(t^{n+1})} \right), X \right)$$

is representable by a scheme $\mathcal{L}_n(X)$. In particular we have $\mathcal{L}_n(X)(F) = X \left(\frac{F[t]}{(t^{n+1})} \right)$ for any field extension F of \mathbb{C} . This scheme is called the scheme of n -jets of X . Remark also that $\mathcal{L}_0(X) = X$.

The truncation morphisms $\pi_m^n : \mathcal{L}_n(X) \rightarrow \mathcal{L}_m(X)$ are affine, where $n \geq m$, and thus we can consider the scheme $\mathcal{L}(X) = \varprojlim_n \mathcal{L}_n(X)$, which is called the arc space of X . It is equipped with projection morphisms $\pi_n : \mathcal{L}(X) \rightarrow \mathcal{L}_n(X)$. A morphism $f : X \rightarrow Y$ of algebraic varieties induces morphisms $f_n : \mathcal{L}_n(X) \rightarrow \mathcal{L}_n(Y)$ and $f : \mathcal{L}(X) \rightarrow \mathcal{L}(Y)$, which are compatible with the projection morphisms.

1.3. Motivic zeta function. We consider

$$\mathfrak{X}_n := \{\varphi \in \mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^d) \mid \text{ord}_t f_n(\varphi) = n, \pi_0^n(\varphi) = 0\}$$

for $n \in \mathbb{N}$. This is a locally closed subset of $\mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^d)$ and thus defines a class $[\mathfrak{X}_n]$ in $K_0(\text{Var}_{\mathbb{C}})$. The (local) motivic zeta function of f (at the origin 0) is then defined as

$$Z_f(T) := \sum_{n>0} [\mathfrak{X}_n] \mathbb{L}^{-dn} T^n \in \mathcal{M}_{\mathbb{C}}[[T]].$$

Our goal now is to incorporate a (regular) differential form ω of maximal degree in this definition. This is done in [2], [3] and [21] with a restriction on the support of ω . In [22] and [18] there is no restriction on the support but only the topological zeta function is used. We will show how we can see the coefficients of the motivic zeta function as the motivic measure of a subset of $\mathcal{L}(\mathbb{A}_{\mathbb{C}}^d)$. Assume that X is a smooth variety of dimension d . Let C be the boolean algebra of cylindrical subsets of $\mathcal{L}(X)$, i.e. subsets of the form $\pi_n^{-1}(A')$, where A' is a constructible subset of $\mathcal{L}_n(X)$ for some $n \in \mathbb{N}$. Recall that for cylindrical subsets A we have that $\pi_n(A) \mathbb{L}^{-dn} \in \mathcal{M}_{\mathbb{C}}$ stabilizes if n tends to infinity; we denote this element by $\mu(A)$. This is called the (naive) motivic measure of A . Define \mathcal{Z}_n to be $\{\varphi \in \mathcal{L}(\mathbb{A}_{\mathbb{C}}^d) \mid \text{ord}_t f(\varphi) = n, \pi_0(\varphi) = 0\}$ and note that $\mu(\mathcal{Z}_n) = [\mathfrak{X}_n] \mathbb{L}^{-dn}$. Hence the coefficients of the motivic zeta function are the measures of \mathcal{Z}_n .

Consider now a regular differential form ω of maximal degree on X . Remark that $\text{div}(\omega)$ is a divisor and consider $\Delta_e = \{\varphi \in \mathcal{L}(X) \mid \text{ord}_t \text{div}(\omega)(\varphi) = e\}$, which is a cylindrical subset. We define

$$\mu_{\omega} : C \rightarrow \hat{\mathcal{M}}_{\mathbb{C}} : A \mapsto \sum_{e \in \mathbb{N}} \mu(A \cap \Delta_e) \mathbb{L}^{-e}$$

and we call this the motivic measure with respect to ω . This sum converges in $\hat{\mathcal{M}}_{\mathbb{C}}$ since there exists an $i \in \mathbb{N}$ such that $\mu(A \cap \Delta_e) \mathbb{L}^{-e} \in F^0$ for all $e \in \mathbb{N}$.

Definition. The (local) motivic zeta function of f with respect to ω (at 0) is

$$Z_{f,\omega}(T) := \sum_{i>0} \mu_{\omega}(\mathcal{Z}_i) T^i \in \hat{\mathcal{M}}_{\mathbb{C}}[[T]].$$

We find that $Z_{f,\omega}(T)$ coincides with $Z_f(T)$ if ω is the standard form $dx_1 \wedge \cdots \wedge dx_n$ and if we consider the coefficients in $\hat{\mathcal{M}}_{\mathbb{C}}$.

We also have a formula in terms of an embedded resolution. Let $\pi : X \rightarrow \mathbb{A}_{\mathbb{C}}^n$ be an embedded resolution of singularities of f and ω , i.e. a proper morphism such that $\pi^{-1}(f^{-1}(0) \cup \text{Supp } \omega)$ is a strict normal crossing divisor and such that π is an isomorphism outside $f^{-1}(0) \cup \text{Supp } \omega$. Let $E_j, j \in J$, be the irreducible components of $\pi^{-1}(f^{-1}(0) \cup \text{Supp } \omega)$, N_j the multiplicity of E_j in $\pi^* f$ and $\nu_j - 1$ the multiplicity of $\pi^* \omega$ along E_j . Remark that $(N_j, \nu_j) \neq (0, 0)$ for all $j \in J$, but $N_j = 0$ is possible for some $j \in J$ since it can happen that $\text{Supp}(\omega) \not\subset f^{-1}(0)$.

Theorem 1.1. Define $E_I = \cap_{i \in I} E_i$ and $E_I^{\circ} = \cap_{i \in I} E_i \setminus (\cup_{i \in J \setminus I} E_i)$ for $I \subset J$. Then we have

$$Z_{f,\omega}(T) = \sum_{\emptyset \neq I \subset J} (\mathbb{L} - 1)^{|I|} [E_I^{\circ} \cap \pi^{-1}(0)] \prod_{i \in I} \frac{T^{N_i}}{\mathbb{L}^{\nu_i} - T^{N_i}} \in \hat{\mathcal{M}}_{\mathbb{C}}[[T]]$$

Proof. Using the techniques of [10, Theorem 2.4] one can show this easily. \square

1.4. Monodromic motivic zeta function. Consider

$$\mathfrak{X}_{n,1} := \{\varphi \in \mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^d) \mid f_n(\varphi) \equiv t^n \pmod{(t^{n+1})}, \pi_0^n(\varphi) = 0\}.$$

for $n \in \mathbb{N}$. This is a closed subset of $\mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^d)$. We have a (natural) μ_n -action defined by $a \cdot \varphi(t) = \varphi(at)$, where $a \in \mu_n$ and $\varphi \in \mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^d)$. Hence we can consider $[\mathfrak{X}_{m,1}]$ in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. The monodromic motivic zeta function of f is then

$$(4) \quad Z_f^{\hat{\mu}}(T) := \sum_{n>0} [\mathfrak{X}_{n,1}] \mathbb{L}^{-dn} T^n \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}[[T]].$$

Look at $\mathcal{Z}_{n,1} = \{\varphi \in \mathcal{L}(\mathbb{A}_{\mathbb{C}}^d) \mid f(\varphi) \equiv t^n \pmod{(t^{n+1})}, \pi_0(\varphi) = 0\}$. This has an action of μ_n like before. Then $\mathcal{Z}_{n,1} \cap \Delta_e$ is a cylindrical subset for every $e \in \mathbb{N}$ and the action induces an action on $\pi_m(\mathcal{Z}_{n,1} \cap \Delta_e)$ for every $m \in \mathbb{N}$. The sequence $\pi_m(\mathcal{Z}_n \cap \Delta_e) \mathbb{L}^{-md}$ stabilizes in $\hat{\mathcal{M}}_{\mathbb{C}}^{\hat{\mu}}$ if m tends to ∞ and we denote this element by $\mu(\mathcal{Z}_{n,1} \cap \Delta_e)$. This leads us to

$$\mu_{\omega}(\mathcal{Z}_{n,1}) = \sum_{e \in \mathbb{N}} \mu(\mathcal{Z}_{n,1} \cap \Delta_e) \mathbb{L}^{-e} \in \hat{\mathcal{M}}_{\mathbb{C}}^{\hat{\mu}}.$$

This sum converges in $\hat{\mathcal{M}}_{\mathbb{C}}^{\hat{\mu}}$ by the same argument as before.

Definition. The (local) monodromic motivic zeta function of f with respect to ω (at the origin) is

$$Z_{f,\omega}^{\hat{\mu}}(T) := \sum_{i \geq 0} \mu_{\omega}(\mathcal{Z}_{i,1}) T^i \in \hat{\mathcal{M}}_{\mathbb{C}}^{\hat{\mu}}[[T]].$$

Remark that this definition coincides again with the one in (4) if ω is the standard form and we consider the coefficients in $\hat{\mathcal{M}}_{\mathbb{C}}^{\hat{\mu}}$.

We also have a formula for this zeta function. Consider again the situation of Theorem 1.1. Suppose $\emptyset \neq I \subseteq J$. Define $m_I = \gcd_{i \in I}(N_i)$. We will introduce \tilde{E}_I° as a unramified Galois cover of E_I° with Galois group μ_{m_I} . Let U be an affine Zariski open such that $f \circ h = uv^{m_I}$, where u is a unit and v a regular function on U . Then the restriction of \tilde{E}_I° above $E_I^{\circ} \cap U$ is defined as

$$\{(z, y) \in \mathbb{A}_{\mathbb{C}}^1 \times U \mid z^{m_I} = u^{-1}\}.$$

Since another choice of u and v induces an isomorphism, these covers glue to a finite Galois cover \tilde{E}_I° of E_I° . The (natural) μ_{m_I} -action is obtained by multiplying the z -coordinates with elements of μ_{m_I} , which gives us an element $[\tilde{E}_I^{\circ}]$ in $K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}})$.

Theorem 1.2. *We have the following equality:*

$$Z_{f,\omega}(T) = \sum_{\emptyset \neq I \subseteq J} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^{\circ} \cap \pi^{-1}(0)] \prod_{i \in I} \frac{T^{N_i}}{\mathbb{L}^{\nu_i} - T^{N_i}} \in \hat{\mathcal{M}}_{\mathbb{C}}^{\hat{\mu}}[[T]].$$

Proof. Using the techniques of [10, Theorem 2.4] one can show this easily. \square

1.5. Topological and other zeta functions. Given a ring morphism $\chi : \hat{\mathcal{M}}_{\mathbb{C}} \rightarrow R$, we can consider the specialization of these motivic zeta functions, i.e.

$$Z_{f,\omega}(T) = \sum_{i \geq 0} \chi(\mu_{\omega}(\mathcal{Z}_n)) T^i \in R[[T]].$$

In this sense we can obtain several other and already known zeta functions such as p -adic zeta functions and Hodge zeta functions.

1.5.1. Topological zeta function. The topological zeta function will be used and thus we discuss this incarnation in more detail. The topological Euler characteristic $\chi_{\text{top}}(X) \in \mathbb{Z}$ of a variety X has the following properties:

- $\chi_{\text{top}}(X) = \chi_{\text{top}}(X \setminus Z) + \chi_{\text{top}}(Z)$ for all varieties X and closed subvarieties Z of X ,
- $\chi_{\text{top}}(X \times Y) = \chi_{\text{top}}(X) \cdot \chi_{\text{top}}(Y)$ for all varieties X and Y ,
- $\chi_{\text{top}}(\mathbb{A}_{\mathbb{C}}^1) = 1$

This implies that we can consider it as a ring morphism $\chi_{\text{top}} : \mathcal{M}_{\mathbb{C}} \rightarrow \mathbb{Z}$. Since $\chi_{\text{top}}(\mathbb{L}) = 1$, we cannot extend it to a morphism from $\hat{\mathcal{M}}_{\mathbb{C}}$. As discussed in [9], we can still apply χ_{top} to elements of $\hat{\mathcal{M}}_{\mathbb{C}}$ which are the image of an element of $\mathcal{M}_{\mathbb{C}}$.

The local topological zeta function of f is defined as the rational function

$$Z_{f,\omega}^{\text{top}}(s) = \sum_{\emptyset \neq I \subseteq J} \chi_{\text{top}}(E_I^{\circ} \cap \pi^{-1}(0)) \prod_{i \in I} \frac{1}{N_i s + \nu_i} \in \mathbb{Q}(s),$$

and the twisted local topological zeta function as

$$Z_{f,\omega}^{\text{top},(e)}(s) = \sum_{\substack{\emptyset \neq I \subseteq J, \\ e|m_I}} \chi_{\text{top}}(E_I^\circ \cap \pi^{-1}(0)) \prod_{i \in I} \frac{1}{N_i s + \nu_i} \in \mathbb{Q}(s),$$

where $e \in \mathbb{N}$.

Given a character α of $\hat{\mu}$, there is a natural ring homomorphism

$$\chi_{\text{top}}(\cdot, \alpha) : \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \rightarrow \mathbb{Z} : X \mapsto \sum_{q \geq 0} \dim H^q(X, \mathbb{C})_\alpha$$

where $H^*(X, \mathbb{C})_\alpha$ is the part of $H^*(X, \mathbb{C})$ on which $\hat{\mu}$ acts by multiplication by α .

Remark that there always exists a character α of given order e and that $\chi_{\text{top}}(X, \alpha)$ only depends on α . We denote this by $\chi_{\text{top}}^{(e)}(X) = \chi_{\text{top}}(X, \alpha)$. We can apply χ_{top} to $Z_{f,\omega}(\mathbb{L}^{-n})$ where $n \in \mathbb{N}$ since it is equal to

$$\sum_{\emptyset \neq I \subseteq J} [E_I^\circ \cap \pi^{-1}(0)] \prod_{i \in I} \frac{(\mathbb{L} - 1) \mathbb{L}^{-n N_i}}{\mathbb{L}^{\nu_i} - \mathbb{L}^{-n N_i}} = \sum_{\emptyset \neq I \subseteq J} (\mathbb{L} - 1)^{|I|} [E_I^\circ \cap \pi^{-1}(0)] \prod_{i \in I} \frac{1}{[\mathbb{P}_{\mathbb{C}}^{N_i n + \nu_i - 1}]}.$$

Another definition of $Z_{f,\omega}^{\text{top}}(s)$ is then the unique rational function such that

$$Z_{f,\omega}^{\text{top}}(n) = \chi_{\text{top}}(Z_{f,\omega}(\mathbb{L}^{-n}))$$

for all $n \in \mathbb{N}$. Analogous we have that $Z_{f,\omega}^{\text{top},(e)}(s)$ is the unique rational function such that

$$Z_{f,\omega}^{\text{top},(e)}(n) = \chi_{\text{top}}^{(e)}((\mathbb{L} - 1) Z_{f,\omega}^{\hat{\mu}}(\mathbb{L}^{-n})).$$

for all $n \in \mathbb{N}$. In this sense the (twisted) topological zeta functions are avatars of the motivic zeta functions.

2. SPLICE DIAGRAMS

This section is dedicated to the notion of splice diagrams as described in [18]. We will not discuss it in full generality but rather stick to the case of plane curve singularities. Consider a polynomial $f \in \mathbb{C}[x, y]$, a (regular) differential 2-form ω on $\mathbb{A}_{\mathbb{C}}^2$ and an embedded resolution of singularities $\pi : X \rightarrow \mathbb{A}_{\mathbb{C}}^2$ for (f, ω) .

The topology of the singularity can be described by means of the dual graph $G = G_\pi$ associated to π , f and ω . Let $E_i, i \in J$, be the irreducible components of $\pi^{-1}(f^{-1}(0) \cup \text{Supp } \omega)$. We have then three types of components: exceptional curves, strict transforms of components of $\{f = 0\}$ and strict transforms of components of $\text{Supp}(\omega)$. This type is not unique, i.e. a component can be a strict transform of a component of $\{f = 0\}$ and a strict transform of a component of $\text{Supp}(\omega)$, but this is the only case where a component can have multiple types.

Each exceptional curve E_i determines a vertex of G and edges correspond to the intersection points of the exceptional curves. Note that this G is a tree, each exceptional curve is rational and $\det(-I(G)) = 1$, where $I(H)$ is the negative definite intersection matrix $(E_i \cdot E_j)_{i,j \in H}$ if H is a subset of the nodes of G . These are exactly the conditions of an integral homology sphere and thus we can use all the machinery developed in [18].

We can now talk about the splice diagram $\Gamma = \Gamma_\pi(f, \omega)$. The underlying graph is the one obtained by removing all nodes of G of valency 2. We add to this graph some decorations. On each pair (v, e) where v is a node of Γ and e is an edge starting at v , we have the decoration $d_{ve} = \det(-I(G_{ve}))$, where G_{ve} is the component of $G \setminus \{v\}$ in the direction of e .

Each irreducible component of the strict transform of $\{f = 0\}$ intersecting in the exceptional component E corresponding to the node v is represented by an arrow a attached at v and has the multiplicity N_a of f along E as a decoration. Similarly a component of a strict transform of $\text{Supp } \omega$ is being displayed by a dotted arrow a and again the multiplicity $\nu_a - 1$ is the associated decoration.

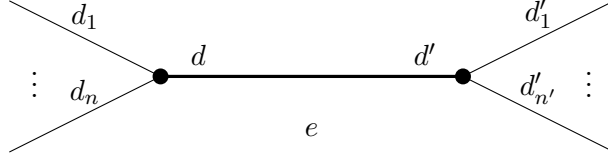
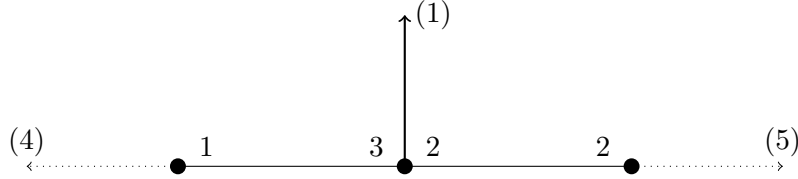


FIGURE 2. Generic situation

It is important to stress that we are only dealing here with plane curve singularities. All these can easily be extended to the case of integral homology spheres as in [18].

Example 2.1. Consider the cusp $f = x^3 - y^2$, its minimal embedded resolution π and differential form $\omega = x^4 y^5 dx \wedge dy$. Then its splice diagram is the following:



We have the following properties of the decorations $\{d_{ve}\}_{e,v}$ of Γ : $d_{ve} \geq 1$, $\{d_{ve}\}_e$ are pairwise coprime for a fixed node v and any edge determinant q_e is positive. Recall that this edge determinant is $\det(-I(G_e))$, where G_e are the exceptional curves of G that lie on e . There is an easier formula. Consider the situation in Figure 2. The edge determinant q_e is then equal to $dd' - DD'$ where $D = \prod_{i=1}^n d_i$ and $D' = \prod_{i=1}^{n'} d'_i$.

It turns out that this splice diagram is very useful for computing the multiplicities of all the exceptional components. We only need the multiplicities at the strict transforms and the decorations on the splice diagram. For a node v of Γ , which corresponds to an exceptional curve, we have

$$(5) \quad N_v = \sum_{a \text{ arrow}} N_a l_{va}$$

where l_{va} is the product of the edge decorations adjacent to the path from v to a but not on it. Analogously, we have

$$(6) \quad \nu_v = \sum_{w \text{ node}} (2 - \delta_w) l_{vw} + \sum_{a \text{ dotted arrow}} l_{va} (\nu_a - 1).$$

This δ_v is the valency of the node v considered without arrows.

2.1. Diagrams. We introduce now what we will call a diagram. While creating splice diagrams, we are deleting nodes of valency two in the graph. But the formulas still work without doing so. Consider now dual graphs of resolutions as before, where we may choose how many valency two nodes we have removed. The decorations are still the same. From now on we will call this a *diagram*.

Definition. We call a diagram Γ *realizable* if it is a dual graph of a resolution of singularities. A diagram with no nodes of valency 2 is called *minimally reduced*.

So a splice diagram is a minimally reduced diagram. Of course every diagram can be reduced to a minimally reduced diagram by deleting all the nodes of valency two.

Lemma 2.2. (1) *If a diagram satisfies $q_e = 1$ for all edges e and $d_{va} = 1$ for all (dotted) arrows a attached at a node v , then it must be realizable.*

(2) *Every reduced diagram can be ‘extended’ or ‘refined’ to a realizable diagram. This can be done by (re)adding nodes (of valency 2) on those edges with $q_e > 1$ and arrows with $d_{va} > 1$.*

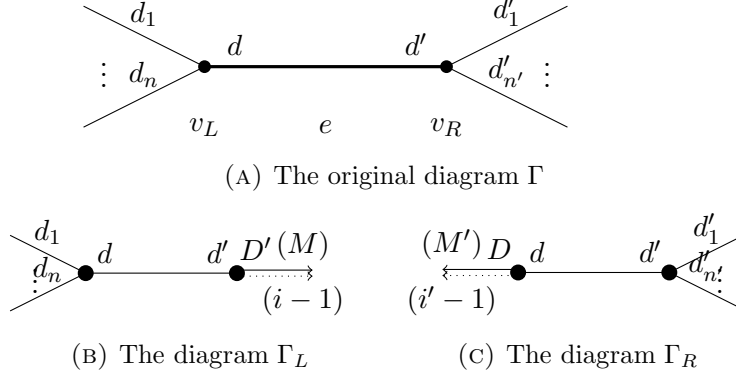


FIGURE 3. Splicing a diagram Γ along an edge e .

- (3) *The possible ways to add nodes to the edge e is not unique and corresponds to smooth refinements of the fan associated to the cone $(D, d)\mathbb{R}_{\geq 0} + (d', D')\mathbb{R}_{\geq 0} \subseteq \mathbb{R}^2$ where $D = \prod_{i=1}^n d_i$ and $D' = \prod_{i=1}^{n'} d'_i$, where we consider the situation as in Figure 2.*

Proof. This follows from interpreting toric concepts described in [6, Chapter 10]. \square

2.2. Zeta functions of diagrams. Recall that we have a formula for our zeta functions in terms of an embedded resolution. In [18] Némethi and Veys define a topological zeta function in terms of the splice diagram. We will give here another definition which is easily seen to be equivalent and is actually just the formula as in Theorem 1.1.

Let Γ be a realizable diagram. We define the *topological zeta function* of Γ to be

$$\begin{aligned}
 Z_{\Gamma}^{\text{top}}(s) &:= \sum_{v \text{ is a node}} \frac{2 - \delta_v}{(N_v s + \nu_v)} \\
 &+ \sum_{e=(v,w) \text{ is an edge}} \frac{1}{(N_v s + \nu_v)(N_w s + \nu_w)} \\
 &+ \sum_{a \text{ (dotted) arrow at } v} \frac{1}{(N_v s + \nu_v)(N_a s + \nu_a)} \in \mathbb{Q}(s).
 \end{aligned}$$

If Γ is not realizable, let Γ' be a realizable refinement of Γ and define $Z_{\Gamma}^{\text{top}}(s) := Z_{\Gamma'}^{\text{top}}(s)$. To prove that this is well-defined, we remark two things:

- If you add a node on a realizable diagram Γ such that the resulting diagram is still realizable, the topological zeta function does not change. This coincides with blowing up in an intersection point of the two exceptional curves corresponding to the adjoining nodes.
- For any diagram, you can go from a realizable refinement to any other realizable refinement by blowing up points and doing the opposite operation.

In what follows we will only give the definition of the considered zeta functions in the case of a realizable diagram since these remarks will also imply that it is well-defined in those cases.

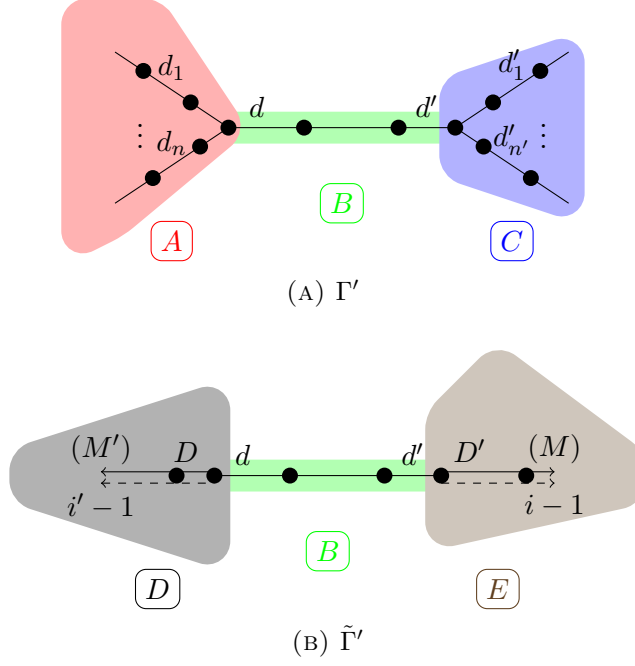


FIGURE 4. The original diagram Γ' and the new diagram $\tilde{\Gamma}'$, used in the proof of Theorem 3.1 and consisting of the same nodes and edges as the diagrams in Figure 5.

Definition. Let Γ be a realizable diagram. We define the *motivic zeta function* of Γ as

$$\begin{aligned}
Z_{\Gamma}(T) &:= \sum_{v \text{ is a node}} \frac{(\mathbb{L} - 1)(\mathbb{L} + 1 - \delta_v)T^{N_v}}{\mathbb{L}^{\nu_v} - T^{N_v}} \\
&+ \sum_{e=(v,w) \text{ is an edge}} \frac{(\mathbb{L} - 1)^2 T^{N_v + N_w}}{(\mathbb{L}^{\nu_v} - T^{N_v})(\mathbb{L}^{\nu_w} - T^{N_w})} \\
&+ \sum_{a \text{ (dotted) arrow at } v} \frac{(\mathbb{L} - 1)^2 T^{N_v + N_a}}{(\mathbb{L}^{\nu_v} - T^{N_v})(\mathbb{L}^{\nu_a} - T^{N_a})} \in \hat{\mathcal{M}}_{\mathbb{C}}[[T]].
\end{aligned}$$

It is possible to define motivic zeta functions for splice diagrams using similar formulas as described in [20]. These formulas for the topological zeta function are used in [18] to prove (2). However it is unlikely that a proof of Theorem 3.1 can be constructed using these formulas without our notion of diagrams because of the complexity of the formulas.

A monodromic motivic zeta function is not available for our notion of diagrams. However we can define a twisted topological zeta function. Consider an $e \in \mathbb{N}$ and define

$$\begin{aligned}
Z_{\Gamma}^{\text{top},(e)}(s) &:= \sum_{\substack{v \text{ is a node,} \\ e|N_v}} \frac{2 - \delta_v}{(sN_v + \nu_v)} \\
&+ \sum_{\substack{e=(v,w) \text{ is an edge,} \\ e|N_v, e|N_w}} \frac{1}{(sN_v + \nu_v)(sN_w + \nu_w)} \\
&+ \sum_{\substack{a \text{ (dotted) arrow at } v, \\ e|N_v, e|N_a}} \frac{1}{(sN_v + \nu_v)(sN_a + \nu_a)} \in \mathbb{Q}(s)
\end{aligned}$$

if Γ is a realizable diagram.

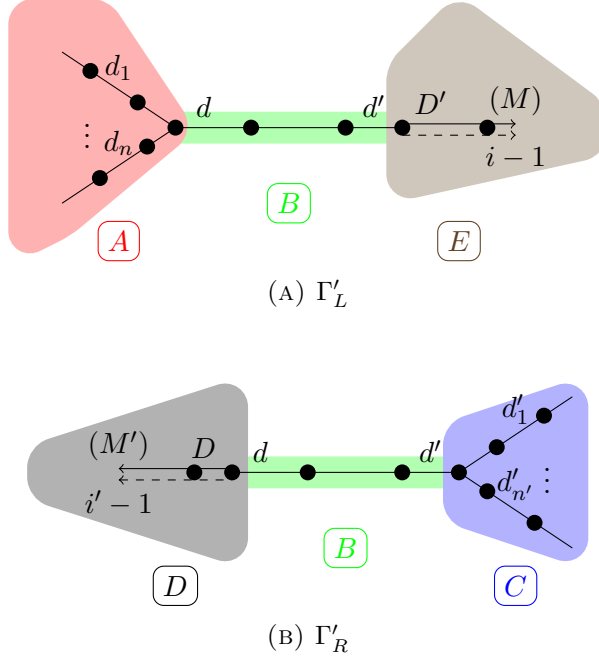


FIGURE 5. The diagrams Γ'_L and Γ'_R , used in the proof of Theorem 3.1 and consisting of the same nodes and edges as the diagrams in Figure 4.

3. SPLICING FORMULA FOR MOTIVIC ZETA FUNCTION

In this section we will prove a splicing formula for motivic zeta functions for (splice) diagrams. This immediately generalizes [18, Theorem 3.2.4 (1)].

Consider a diagram Γ with an edge e between v_L to v_R as in Figure 3(A). Recall that $D = \prod_{i=1}^n d_i$ and $D' = \prod_{i=1}^{n'} d'_i$. We define the multiplicities

$$(7) \quad M = \sum_{a \text{ arrow at right side}} N_a l_{ea} \quad \text{and} \quad M' = \sum_{a \text{ arrow at left side}} N_a l_{ea},$$

and the multiplicities

$$(8) \quad i = \sum_{w \text{ node at right side}} (2 - \delta_w) l_{ew} + \sum_{a \text{ dotted arrow at right side}} l_{ea} (i_a - 1)$$

and

$$(9) \quad i' = \sum_{w \text{ node at left side}} (2 - \delta_w) l_{ew} + \sum_{a \text{ dotted arrow at left side}} l_{ea} (i_a - 1),$$

where l_{ea} is the product of the edge decorations adjacent to the path from e to a but not on it. This product does not use the decorations on e itself. We obtain the diagram Γ_L by removing all the nodes and edges at the side of v_R and add two arrows and a node as in Figure 3(B). Analogously we have Γ_R as in Figure 3(c). This procedure is called splicing (of Γ along e).

We now state and prove our result.

Theorem 3.1. *Consider a diagram Γ and the splicing of Γ into Γ_L and Γ_R . Then we have*

$$Z_\Gamma(T) = Z_{\Gamma_L}(T) + Z_{\Gamma_R}(T) - \frac{(\mathbb{L} - 1)^2 T^{M+M'}}{(\mathbb{L}^i - T^M)(\mathbb{L}^{i'} - T^{M'})}.$$

Proof. Define the diagram $\tilde{\Gamma}$ as follows:

$$\begin{array}{ccccccc} (M') & D & d & & d' & D' & (M) \\ \leftarrow \text{dotted arrow} & \bullet & \xrightarrow{\tilde{e}} & \bullet & \xleftarrow{\text{dotted arrow}} & & \\ i' - 1 & & & & & & i - 1 \end{array}$$

where \tilde{e} is the edge between the nodes. Remark that this is the dual graph of a (toric) resolution of the polynomial $f = x^M y^{M'}$ and the differential form $\omega = x^{i-1} y^{i'-1} dx \wedge dy$. Hence $Z_{\tilde{\Gamma}}(T) = \frac{(\mathbb{L}-1)^2 T^{M+M'}}{(\mathbb{L}^i - T^M)(\mathbb{L}^{i'} - T^{M'})}$ and thus we need to show that

$$(10) \quad Z_{\Gamma}(T) + Z_{\tilde{\Gamma}}(T) = Z_{\Gamma_L}(T) + Z_{\Gamma_R}(T).$$

We prove this by considering suitable realizable refinements of these diagrams. First, take a realizable refinement Γ' of Γ . This is drawn in Figure 4(A), where you have the division into parts (A), (B) and (C). Remark that the edges crossing the border belong to (B).

Second, take a realizable refinement $\tilde{\Gamma}'$ of $\tilde{\Gamma}$ such that the edge e of Γ has the same refinement as the edge \tilde{e} of $\tilde{\Gamma}$. Hence we can consider Figure 4(B) where we have a division into (D), (B) and (E) where the subdiagram (B) is the same as in Figure 4(A).

We glue these refinements together to obtain refinements Γ'_L and Γ'_R of Γ_L and Γ_R . This is shown in Figures 5(A) and 5(B).

We compare the diagrams Γ' and $\tilde{\Gamma}'$ to the diagrams Γ'_R and Γ'_L . Both groups contain the subdiagrams (A), (B), (B), (C), (D) and (E). Hence if we calculate the motivic zeta function, where we take the sum over all nodes and edges in these subdiagrams, we have proven (10) if we show that the corresponding nodes have the same multiplicities. But this is easily seen by using (5), (6), (7), (8) and (9). \square

We can connect our proof to the splicing of links. Recall that splicing consists of taking two links, where in each a knot is selected, removing a tubular neighborhood around these knots and glueing the remainders together. However if you glue these removed tubular neighbourhoods together, you find the link of Γ' .

4. ALGEBRAIC DEPENDENCE OF THE MONODROMIC MOTIVIC ZETA FUNCTION

Following Theorem 3.1, we want to define a monodromic zeta function for a diagram. But it turns out that this is not possible. Consider for this the family of polynomials

$$f_{\lambda} = xy^2(x-y)(x-\lambda y) \in \mathbb{C}[x, y],$$

where $\lambda \in \mathbb{C} \setminus \{0, 1\}$. The splice diagram associated to this family of polynomials is independent of λ . But we do have the following result.

Proposition 4.1. *There exist $\lambda, \lambda' \in \mathbb{C} \setminus \{0, 1\}$ such that*

$$Z_{f_{\lambda}}^{\hat{\mu}}(T) \neq Z_{f_{\lambda'}}^{\hat{\mu}}(T).$$

This results shows that a monodromic motivic zeta function cannot be defined for a diagram since the diagrams for this family of polynomials are the same.

- Define the Grothendieck group of abelian varieties $A_{\mathbb{C}}$ as the abelian group generated by isomorphism classes of abelian varieties with the relations $[A \oplus B] = [A] + [B]$ and
- define $A_{\mathbb{C}}^0$ as the abelian group generated by isogeny classes of abelian varieties and relations $[A \oplus B] = [A] + [B]$.

The structure of $A_{\mathbb{C}}^0$ is easier since Poincaré's complete irreducibility theorem [17, p. 173] implies that $A_{\mathbb{C}}^0$ is isomorphic to the free abelian group on simple abelian varieties. Hence equality of the classes of two abelian varieties in $A_{\mathbb{C}}^0$ implies that they are isogenous.

In the appendix we describe a group morphism $\widetilde{\text{Pic}} : \hat{\mathcal{M}}_{\mathbb{C}} \rightarrow A_{\mathbb{C}}$. This morphism sends the class of a smooth complete variety to the class of its Jacobian. We will use this morphism in the following proof, where we compose it with the forgetful morphism $A_{\mathbb{C}} \rightarrow A_{\mathbb{C}}^0$.

Proof of Proposition 4.1. Let $\pi : X \rightarrow \mathbb{A}_{\mathbb{C}}^2$ be the blowup at the origin. This is an embedded resolution of singularities for f_{λ} . The formula of Denef-Loeser then tells us that we need to prove that the class of

$$\tilde{E}_{\lambda}^{\circ} = \{(x, y) \in \mathbb{C}^2 \mid y^5 = x(x-1)(x-\lambda), y \neq 0\}$$

does depend on λ .

We will now prove that if two varieties are equal in $\hat{\mathcal{M}}_{\mathbb{C}}$, then their jacobians are isogenous. Combining this with [7, Proposition 2.7] we find that the set $\{[\tilde{E}_{\lambda}^{\circ}] \in \hat{\mathcal{M}}_{\mathbb{C}} \mid \lambda \in \mathbb{C} \setminus \{0, 1\}\}$ is infinite.

We have a map $\hat{\mathcal{M}}_{\mathbb{C}} \xrightarrow{\text{Pic}} A_{\mathbb{C}} \rightarrow A_{\mathbb{C}}^0$ which sends a complete smooth variety to the class of its jacobian. Using the remark before the start of the proof, we find that equality in $\mathcal{M}_{\mathbb{C}}$ implies that their jacobians are isogenous. \square

This has implications on how to formulate a splicing formula for the monodromic motivic zeta functions. The same proof of Theorem 3.1 will work if you work with a more general notion of a diagram. In this generalization you need to encode the information of \tilde{E}_I° more carefully, for example by remembering the locations of the branch points of the cover $\tilde{E}_I^{\circ} \rightarrow \mathbb{P}_{\mathbb{C}}^1$.

Another way is to formulate the splicing formula in terms of f and ω .

Theorem 4.2. *Let $f \in \mathbb{C}[x, y]$ and let ω be a differential form. Also fix an edge e in an associated diagram Γ for some embedded resolution and denote by Γ_L and Γ_R the resulting diagrams after splicing along e . Then there exists $f_L, f_R \in \mathbb{C}[x, y]$ and differential forms ω_L, ω_R , whose splice diagrams are Γ_L and Γ_R for some embedded resolutions, such that*

$$Z_{f, \omega}^{\hat{\mu}}(T) = Z_{f_1, \omega_1}^{\hat{\mu}}(T) + Z_{f_2, \omega_2}^{\hat{\mu}}(T) - \frac{(\mathbb{L} - 1)^2 T^{M+M'}}{(\mathbb{L}^i - T^M)(\mathbb{L}^{i'} - T^{M'})}.$$

5. MONODROMY CONJECTURE

In this section we discuss how we can lift the results in [18] about the generalized monodromy conjecture to the motivic level. Recall that we consider our polynomial f as a germ $(\mathbb{A}_{\mathbb{C}}^2, 0) \rightarrow (\mathbb{A}_{\mathbb{C}}^1, 0)$. Let F_0 be the Milnor fiber of this germ, $h_i : H_i(F_0, \mathbb{C}) \rightarrow H_i(F_0, \mathbb{C})$ the algebraic monodromy ($i = 0, 1$), $\Delta_i = \det(tI - h_i)$ the characteristic polynomial of h_i and $\zeta(t) = \Delta_1/\Delta_0$.

Given a diagram Γ , we define the formula

$$\zeta_{\Gamma}(t) = \prod_{v \text{ is a node}} (t^{N_v} - 1)^{\delta'_v}$$

where δ'_v is the valency of v considered in the graph including the arrows corresponding to strict transforms of f , but without the dotted arrows corresponding to the strict transforms of $\text{Supp } \omega$. This recovers the monodromy zeta function of f . Remark that $\Delta_0 = t^d - 1$ where $d = \gcd_{a \text{ is an arrow}} N_a$, and thus we can discuss monodromy eigenvalues of a diagram.

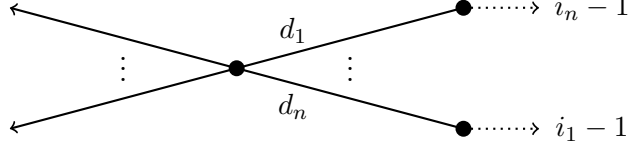
In what follows it will be implicit that monodromy eigenvalues are the monodromy eigenvalues of our fixed f .

5.1. Motivic zeta function. Némethi and Veys defined a differential form ω on a diagram Γ to be allowed if the following conditions are satisfied:

- $(N_a, i_a) \neq (0, 0)$ for all (dotted) arrows.
- each star-shaped subdiagram with the induced decorations obtained after repeated splicing needs to be allowed.
- if Γ is a star-shaped diagram the following condition needs to be satisfied:

Let the central node be connected to n boundary vertices whose supporting edges have decorations $\{d_l\}_{l=1}^n$, and with r other incident edges connecting with arrowheads. Then the decorations $i_1 - 1, \dots, i_n - 1$ of the dashed arrows at these boundary vertices are subject to the following restrictions:

If $d_l | i_l$ for at least $n + r - 2$ indices $l \in \{1, \dots, n\}$, then $i_l = d_l$ for at least $n + r - 2$ of the indices l .



r arrowheads
which might be doublearrows

n boundary vertices

Némethi and Veys showed that this set of allowed forms has the following properties:

- for every allowed form ω and every pole s_0 of $Z_{f,\omega}^{\text{top}}(s)$, $\exp(2\pi i s_0)$ is a monodromy eigenvalue.
- $dx \wedge dy$ is allowed;
- for every monodromy eigenvalue λ , there is an allowed form ω such that $Z_{f,\omega}^{\text{top}}(s)$ has a pole s_0 and $\lambda = \exp(2\pi i s_0)$.

To state the generalized monodromy conjecture for the motivic zeta function we need the notion of a pole. This has been done by Rodrigues and Veys in [19]. We will use a more direct approach.

Definition. We call $s \in \mathbb{Q}$ a pole of $Z_{f,\omega}(T)$ if there exists no set $U \subseteq \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that

$$Z_{f,\omega}(T) \in \hat{\mathcal{M}}_{\mathbb{C}} \left[\frac{T^N}{\mathbb{L}^\nu - T^N} \right]_{(v,N) \in U} \subseteq \hat{\mathcal{M}}_{\mathbb{C}}[[T]].$$

and such that $s \neq -\frac{\nu}{N}$ for all $(\nu, N) \in U$.

Because $\hat{\mathcal{M}}_{\mathbb{C}}$ is not a domain [13], we are careful in formulating the next theorem.

Theorem 5.1. *Let $f \in \mathbb{C}[x, y]$ and ω an allowed form. Then*

$$Z_{f,\omega}(T) \in \hat{\mathcal{M}}_{\mathbb{C}} \left[\frac{T^N}{\mathbb{L}^\nu - T^N} \right]_{(v,N) \in S} \subseteq \hat{\mathcal{M}}_{\mathbb{C}}[[T]].$$

where $S = \{(\nu, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid (\nu, N) \neq (0, 0), \exp(-2\pi i \frac{\nu}{N}) \text{ is a monodromy eigenvalue of } f\}$.

Proof. The proof goes as in the case of the topological zeta function, where you need first to consider star-shaped realizable diagrams. In this case you need to prove that $\alpha_i = -N_i \frac{\nu}{N} + \nu_i = 1$ for sufficiently many i .

The general case can be proved in the same way as in the proof for the topological zeta function. \square

By specializing to the topological zeta function, we easily find that every monodromy eigenvalue is obtained as a pole. Hence we have proven the generalized monodromy conjecture for the motivic zeta function.

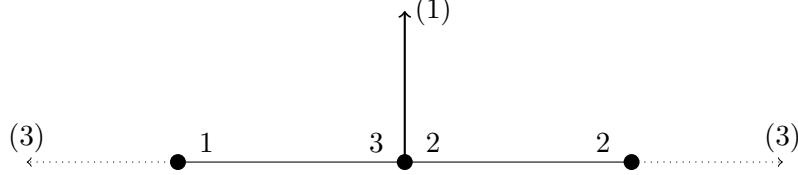
Corollary 5.2. *Consider the set of allowed forms for a diagram Γ of $f \in \mathbb{C}[x, y]$. It satisfies the following conditions:*

- for every allowed form ω , every pole of $Z_{f,\omega}(T)$ induces a monodromy eigenvalue of f . More specifically Theorem 5.1 holds.
- $dx \wedge dy$ is allowed;
- every monodromy eigenvalue is obtained as a pole of the motivic zeta function of f with respect to ω .

5.2. Monodromic motivic zeta function. Theorem 5.1 however does not generalize to the case of the monodromic motivic zeta function. This is due to the fact that an allowed form is made such that the residues of candidate ‘bad’ poles of the topological zeta functions are zero. We give here examples of twisted topological zeta functions because a pole of a twisted topological zeta function will also be a pole of the monodromic motivic zeta function.

We give an example where the twisted topological zeta function has a pole which does not induce a monodromy eigenvalue, and thus an analogue of Corollary 5.2 cannot hold.

Example 5.3. Consider the cusp $f = x^3 - y^2$, the minimal resolution π and differential form $\omega = x^3 y^3 dx \wedge dy$. Then its splice diagram is the following:



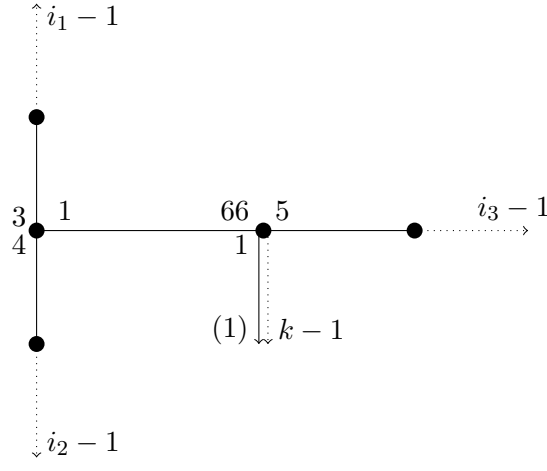
This form ω is allowed but we find that

$$Z_{f,\omega}^{\text{top},(6)}(s) = -\frac{1}{6s + 21},$$

which has $s_0 = -\frac{7}{2}$ as a pole. But $\exp(2\pi i s_0) = -1$ is not a monodromy eigenvalue.

One can wonder if a suitable subset of allowed forms can achieve Corollary 5.2. The following example shows however that we cannot obtain all poles.

Example 5.4. Consider the polynomial $f = (y^3 - x^4)^5 + x^2 y^{15}$. We have the following diagram Γ



where we are already considering some form with $i_1, i_2, i_3, k \in \mathbb{Z}$. The monodromy zeta function is

$$\zeta(T) = \frac{(T^{330} - 1)(T^{60} - 1)}{(T^{66} - 1)(T^{15} - 1)(T^{20} - 1)},$$

which implies that $\lambda = \exp(2\pi i \frac{1}{110})$ is a monodromy eigenvalue. However there exist no i_1, i_2, i_3, k such that $Z_{\Gamma}^{\text{top},(330)}(s)$ has a pole s_0 with the condition $\lambda = \exp(2\pi i s_0)$ and such that $\lambda' = \exp(2\pi i s'_0)$ is a monodromy eigenvalue whenever s'_0 is the pole of $Z_{\Gamma}^{\text{top},(60)}(s)$. Indeed, if λ is a pole, then s_0 needs to be a pole of $Z_{\Gamma}^{\text{top},(330)}(s)$. This implies that $2i_1 + 3i_2 \equiv 3 \pmod{6}$. But now the pole of $Z_{\Gamma}^{\text{top},(60)}(s)$ will not induce a monodromy eigenvalue since $i_1 \equiv 0 \pmod{3}$ and $i_2 \equiv 1 \pmod{2}$.

APPENDIX A. EXISTENCE OF THE PICARD MORPHISM

It turns out that the class of a smooth and proper algebraic variety in the Grothendieck ring of varieties determines the class of its Picard scheme. In this appendix we will define and prove this statement using the argument described in [13]. We do this since [13] was never published and to clarify several steps in his argument.

Let X be a smooth and proper algebraic variety over \mathbb{C} . Recall that the Picard functor $\text{Pic}_{X/\mathbb{C}}$ is a functor from the category of locally Noetherian \mathbb{C} -schemes to the category of abelian groups defined by the formula

$$\text{Pic}_{X/\mathbb{C}}(T) = \text{Pic}(X_T) / \text{Pic}(T)$$

where $X_T = X \times_{\mathbb{C}} T$. It turns out that the associated fppf sheaf is representable by an abelian scheme whose identity component is an abelian variety and whose group of geometric components is finitely generated. This scheme will be denoted by $\text{Pic}(X)$. See [14, Part 5] for more information on the Picard scheme.

Definition. Define $\mathcal{A}_{\mathbb{C}}$ as the abelian group generated by isomorphism classes of commutative algebraic group schemes over \mathbb{C} whose identity component is an abelian variety and whose group of geometric components is finitely generated, subject to the relations $[A \oplus B] = [A] + [B]$.

This leads us to the main result.

Theorem A.1. *There is a (unique) group homomorphism $\text{Pic} : K_0(\text{Var}_{\mathbb{C}}) \rightarrow \mathcal{A}_{\mathbb{C}}$ such that $\text{Pic}([X]) = [\text{Pic}(X)]$ for every smooth proper variety X , and it extends to a morphism $\text{Pic} : \hat{\mathcal{M}}_{\mathbb{C}} \rightarrow \mathcal{A}_{\mathbb{C}}$.*

This theorem provides us with a new technique to compare elements in the Grothendieck ring.

Recall that the Grothendieck group of abelian varieties $A_{\mathbb{C}}$ is defined as the abelian group generated by isomorphism classes of abelian varieties with the relations $[A \oplus B] = [A] + [B]$. Using this theorem, we find the existence of $\widetilde{\text{Pic}} : \hat{\mathcal{M}}_{\mathbb{C}} \rightarrow A_{\mathbb{C}}$ which sends a smooth complete variety to the class of its jacobian. It is obtained by composing Pic and the morphism $\mathcal{A}_{\mathbb{C}} \rightarrow A_{\mathbb{C}}$, where this last map is defined by sending the class of a commutative algebraic group scheme (whose identity component is an abelian variety) to the class of its identity component.

The keystone of the proof is Bittner's presentation of $K_0(\text{Var}_{\mathbb{C}})$, which we restate here.

Theorem. [4, Theorem 3.1] *The Grothendieck group of \mathbb{C} -varieties $K_0(\text{Var}_{\mathbb{C}})$ is isomorphic to the abelian group generated by the isomorphism classes of smooth projective \mathbb{C} -varieties subject to the relations $[\emptyset] = 0$ and $[\text{Bl}_Y X] - [E] = [X] - [Y]$, where X is smooth and projective, $Y \subset X$ is a closed smooth subvariety, $\text{Bl}_Y X$ is the blow-up of X along Y and E is the exceptional divisor of this blow-up.*

This implies the following presentation of $\mathcal{M}_{\mathbb{C}} = K_0(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$:

$\mathcal{M}_{\mathbb{C}}$ is generated by the elements $\frac{[X]}{\mathbb{L}^n}$ where X is smooth and projective, subject to the relations

$$\frac{[X \times \mathbb{P}_{\mathbb{C}}^1]}{\mathbb{L}^{n+1}} = \frac{[X]}{\mathbb{L}^n} + \frac{[X]}{\mathbb{L}^{n+1}},$$

where X is smooth and projective, and the relations $\frac{[\text{Bl}_Y X]}{\mathbb{L}^n} - \frac{[E]}{\mathbb{L}^n} = \frac{[X]}{\mathbb{L}^n} - \frac{[Y]}{\mathbb{L}^n}$, where X , Y and E are as in the theorem.

Proof of Theorem A.1. • We first show that the morphism defined by $\text{Pic}([X]) = [\text{Pic}(X)]$ for X smooth and proper is well-defined as a map from $K_0(\text{Var}_{\mathbb{C}})$ to $\mathcal{A}_{\mathbb{C}}$. Using the Bittner representation, we need to show that

$$\text{Pic}(\text{Bl}_Y X) \oplus \text{Pic}(Y) = \text{Pic}(X) \oplus \text{Pic}(E)$$

where X , Y and E are as in the theorem. Hence we need to show this on the level of associated fppf sheaves. But a blow-up is preserved under base change by a flat morphism and thus $\text{Pic}(\text{Bl}_{Y_T} X_T) = \text{Pic}(X_T) \oplus \mathbb{Z}$ and $\text{Pic}(E_T) = \text{Pic}(Y_T) \oplus \mathbb{Z}$ for any flat morphism $T \rightarrow \mathbb{C}$ which induces the wanted isomorphism [16, Exercises 7.9 and 8.5 on pages 170 and 188].

- The next step is to define $\text{Pic}' : \mathcal{M}_{\mathbb{C}} \rightarrow \mathcal{A}_{\mathbb{C}}$ and prove that this is actually an extension of Pic .

Define $\text{Pic}'([X]/\mathbb{L}^n)$ as

$$A_X^{0,n} \oplus A_X^{c,n}$$

for a projective and smooth variety X and $n \in \mathbb{N}$, where

- $A_X^{c,n}$ is the inverse image of the classes of type $(n+1, n+1)$ under the map $H^{2n+2}(X, \mathbb{Z}) \rightarrow H^{2n+2}(X, \mathbb{C})$ and
- $A_X^{0,n}$ is the Weil intermediate Jacobian associated to the Hodge structure on $H^{2n+1}(X, \mathbb{Z})$. See [24] for more information.

As discussed, we need to verify the two types of relations. Consider X to be a projective smooth \mathbb{C} -variety.

- First we verify the blow-up relations. This is a consequence of [23, Theorem 7.31] and its proof since it induces that

$$H^k(X, \mathbb{Z}) \oplus \left(\bigoplus_{i=0}^{r-2} H^{k-2i-2}(Y, \mathbb{Z}) \right) \oplus H^k(Y, \mathbb{Z})$$

is both isomorphic to $H^k(\text{Bl}_Y X, \mathbb{Z}) \oplus H^k(Y, \mathbb{Z})$ and $H^k(X, \mathbb{Z}) \oplus H^k(E, \mathbb{Z})$ as Hodge structures, where r is the codimension of Y in X .

- Remark that the cup-products map [23, Theorem 11.38] for X and $\mathbb{P}_{\mathbb{C}}^1$ induces

$$H^{i+2}(X \times \mathbb{P}_{\mathbb{C}}^1, \mathbb{Z}) \cong H^{i+2}(X, \mathbb{Z}) \oplus H^i(X, \mathbb{Z})$$

as Hodge structures [11, p. 32].

Since the relations hold on the level of Hodge structures, they also hold for Pic' and thus Pic' is well-defined.

- We show now that Pic' is indeed an extension of Pic and thus we will show that $\text{Pic}(X) \cong A_X^{0,0} \oplus A_X^{c,0}$. Remark that the connected component $\text{Pic}^0(X)$ is the classical Jacobian of X and thus isomorphic to $A_X^{0,0}$.

The Néron-Severi group $\text{NS}(X)$ is defined by the short exact sequence

$$0 \rightarrow \text{Pic}^0(X) \rightarrow \text{Pic}(X) \rightarrow \text{NS}(X) \rightarrow 0$$

and is the group of components of $\text{Pic}(X)$. This group can also be identified with the image of $d : H^1(X, \mathcal{O}_X^\times) \rightarrow H^2(X, \mathbb{Z})$. Now Lefschetz's theorem on $(1, 1)$ -classes [15, p.163] implies that $\text{NS}(X)$ is exactly $A_X^{c,0}$.

Since we are working over an algebraically closed field and $\text{NS}(X)$ is a discrete finitely generated abelian group, the short exact sequence splits and thus $\text{Pic}(X) \cong \text{Pic}^0(X) \oplus \text{NS}(X)$.

- To conclude we remark that $H^n(X, \mathbb{Z}) = 0$ if $n > 2 \dim X$ and thus $\text{Pic}' \frac{[X]}{\mathbb{L}^n} = 0$ if $\dim(X) - n \leq 0$. This implies that Pic' sends every element of F^0 to 0 and thus Pic' can be extended to $\hat{\mathcal{M}}_{\mathbb{C}}$. \square

One of the results Ekedahl obtains with this is the fact that $\hat{\mathcal{M}}_{\mathbb{C}}$ is not a domain.

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